International Journal of Theoretical Physics, Vol. 36, No. 9, 1997

Formal Existence and Uniqueness of the Reichenbachian Common Cause on Hilbert Lattices

Gábor Hofer-Szabó[1]

Received February 7, 1997

The Reichenbachian definition of the common cause for two correlating events is formally generalized for the quantum case in two different ways. It is shown that (1) in the first quantum case, unlike in the classical case, there exists a common cause for any two correlating events, and (2) the common cause is not unique either in the classical or in both quantum cases.

1. INTRODUCTION

If there is a correlation between two events, one possible explanation is that the correlation stems from a common cause. Reichenbach defines this common cause in a probabilistic way and gives some examples for correlation which can be explained by a common cause (Reichenbach, 1956). In Section 2 we present this classical definition, and show a simple example for a Boolean lattice in which correlation occurs between two events without a common cause. In Sections 3 and 4 we generalize the original definition for the quantum case in two different ways according to two possible definitions of the conditional probability on a Hilbert lattice, and we show that in the first case if there is a correlation between two events, then there is a common cause for this correlation. From the technique of the proof it can be also seen that the common cause is not in general unique. In the second case we cannot claim a similar proposition regarding the existence; but we show that neither definition leads to a unique common cause.

[1] Department of Philosophy, Technical University of Budapest, H-1111 Budapest, Müegyetem rkp 3. K. I/59, Hungary; e-mail: gszabo@hps.elte.hu.

In this paper we just concentrate on the mathematical structure of the Reichenbachian definition and we do not investigate the philosophical consequences of these results. In our previous paper (Hofer-Szabó, 1997a) we investigated the Reichenbachian definition regarding its consistency; we analyze the philosophical meaning of the three properties of existence, uniqueness, and consistency of the Reichenbachian common cause together in our next paper (Hofer-Szabó, 1997b).

2. THE CLASSICAL CASE

Let (i) (Ω, F, p) be a Kolmogorovian probability measure space and let (ii) the conditional probability of E given F be defined as usual by

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

Let $A, B \in \Omega$ be two correlated events, i.e.,

$$p(A \cap B) > p(A)p(B) \tag{1}$$

Reichenbach defines the common cause of the correlation as follows:

Definition 1. An event C is said to be the *common cause* of the correlation between A and B if the events A, B, and C satisfy the following relations:

$$p(A \cap B|C) = p(A|C)p(B|C) \tag{2}$$

$$p(A \cap B|\overline{C}) = p\,(A|\overline{C})p(B|\overline{C}) \tag{3}$$

$$p(A|C) > p(A|\overline{C}) \tag{4}$$

$$p(B|C) > p(B|\overline{C}) \tag{5}$$

We denote by $p(\cdot|C)$ and $p(\cdot|\overline{C})$ the probabilities conditioned on C and non-C, respectively. We wish to raise the question of whether a correlation is always explainable by a common cause. We answer this question by showing a simple Boolean lattice where there is a correlation without a common cause. Let the lattice be generated by two elements A and B, and let there be a suitable probabilistic measure on the lattice. (See Fig. 1; we denote the meets by "&"; the measure of the elements is given in parentheses.)

Since $0.2 = p(A \cap B) > p(A)p(B) = 0.16$, there is a correlation between A and B. But there is no element in the lattice which would satisfy (2)–(5) and so could be regarded as the common cause of this correlation. So Boolean lattices do not automatically supply a common cause for every correlation.

It is another question whether we can extend the lattice so that we get a common cause for a given correlation in the lattice. If, for instance, we

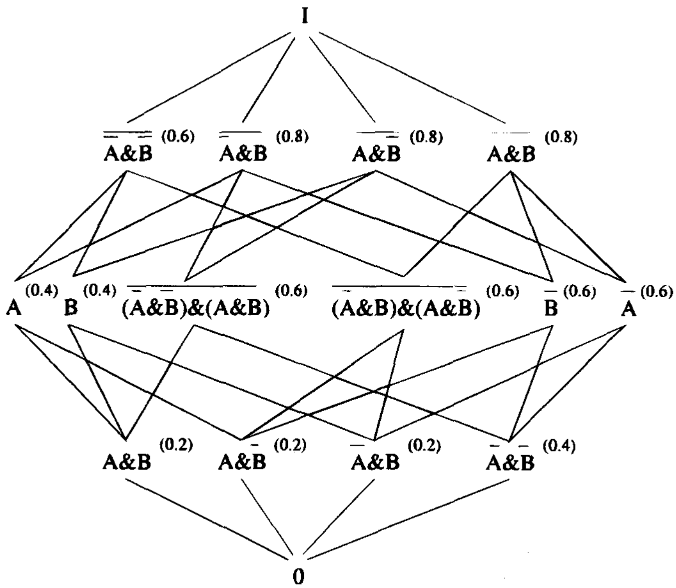

Fig. 1. A Boolean lattice without common cause.

extend the lattice to contain all the Lebesgue-measurable sets in [0, 1], then we can show in a straightforward way that there exist uncountably many common causes of the correlation above. So the classical common cause is in general not unique.

3. FIRST GENERALIZATION

Let (i) $P(H)$ be a Hilbert lattice and W be a pure state represented by the unit vector w. For the projections E and F in the lattice let (ii) the conditional probability of E given F in a state W be defined in the following way:

$$p_w(E|F) = \frac{p_w(E \wedge F)}{p_w(F)} = \frac{\mathrm{Tr}(W(E \wedge F))}{\mathrm{Tr}(WF)}$$

[Now we disregard the logical and mathematical difficulties arising from this generalization of the Bayes rule—that, for example, $p_w(E|F) + p_w(E^{\perp}|F) \neq 1$. This definition can be regarded as a connecting link between the classical and the quantum conditional probabilities.]

Let A, $B \in P$ (H) and assume a correlation between A and B in the state W, i.e.,

$$p_w(A \wedge B) > p_w(A)p_w(B) \tag{6}$$

We now define the common cause of the correlation in the quantum case:

Definition 2. An event C is said to be the *common cause* of the correlation between A and B if the events A, B, and C satisfy the following relations:

$$p_w(A \wedge B|C) = p_w(A|C)p_w(B|C) \tag{7}$$

$$p_w(A \wedge B|C^\perp) = p_w(A|C^\perp)p_w(B|C^\perp) \tag{8}$$

$$p_w(A|C) > p_w(A|C^\perp) \tag{9}$$

$$p_w(B|C) > p_w(B|C^\perp) \tag{10}$$

Now we ask a similar question to the classical case: is there a common cause for every correlation? We claim that the answer is yes.

Theorem 1. Let A and B be elements of a Hilbert lattice, and let them satisfy (6). Then there always exists a third event C in the lattice such that the events A, B, and C satisfy (7)–(10).

Proof. Let A and B be two correlated events in the pure state W, i.e., let $p_w(A \wedge B) > p_w(A)p_w(B)$. Then the following three relations hold: (i) $A \wedge B \neq 0$, (ii) $w \notin A \wedge B$, (iii) $w \notin A, B$.

Now let $\{e, f, g_i\}_{i=3}^N$ be an orthonormal basis in H, so that $e \in A \wedge B$, and $f \in \{e \vee w\}$, where $\{e \vee w\}$ is a two-dimensional subspace in H (since $w \notin A \wedge B$). (We denote by $e, f, \ldots$ both the vectors and the projections.) Now we claim that for the choice $C = \{e \vee g_3 \vee \ldots \vee g_N\}$, C satisfies (7)–(10). Let us see the different conditional probabilities.

By the calculation of the Tr-function in the conditional probabilities $p_w(\cdot|C)$ we use the basis defined above:

$$p_w(A|C) = \frac{\mathrm{Tr}(W(A \wedge C))}{\mathrm{Tr}(WC)}$$

$$= \frac{\langle e, We\rangle + \sum_{i,j=3}^{N,N} a_{ij}(A, C)\langle g_i, Wg_j\rangle}{\langle e, We\rangle + \sum_{i=3}^{N} \langle g_i, Wg_i\rangle} = 1$$

$$p_w(B|C) = \frac{\mathrm{Tr}(W(B \wedge C))}{\mathrm{Tr}(WC)}$$

$$= \frac{\langle e, We\rangle + \sum_{i,j=3}^{N,N} b_{ij}(B, C)\langle g_i, Wg_j\rangle}{\langle e, We\rangle + \sum_{i=3}^{N} \langle g_i, Wg_j\rangle} = 1$$

$$p_w(A \wedge B|C) = \frac{\mathrm{Tr}(W(A \wedge B \wedge C))}{\mathrm{Tr}(WC)}$$

$$= \frac{\langle e, We\rangle + \sum_{i,j=3}^{N,N} c_{ij}(A \wedge B, C)\, \langle g_i, Wg_j\rangle}{\langle e, We\rangle + \sum_{i=3}^{N} \langle g_i, Wg\rangle} = 1$$

where $a_{ij}(A, C)$, $b_{ij}(B, C)$, and $c_{ij}(A \wedge B, C)$ are coefficients depending on the projections in the parentheses. All the conditional probabilities equal 1, since for every $i, j = 1, \ldots, N$, the product $\langle g_i, Wg_j\rangle = 0$.

Now let us see the $C^\perp$ case. Because of (iii), f cannot be an element of A, B, and $A \wedge B$. Since $e \in A, B$ and $A \wedge B$, if f were in A, B, and $A \wedge B$, then w would also be in A, B, and $A \wedge B$, which contradicts (iii). Since $f \notin A, B$ and $A \wedge B$, so $A \wedge C^\perp$, $B \wedge C^\perp$, and $A \wedge B \wedge C^\perp$ are 0-projections. So the probabilities conditioned on $C^\perp$ are equal to 0:

$$p_w(A|C^\perp) = \frac{p_w(A \wedge C^\perp)}{p_w(C^\perp)} = \frac{\mathrm{Tr}(W(A \wedge C^\perp))}{\mathrm{Tr}(WC^\perp)} = 0$$

$$p_w(B|C^\perp) = \frac{p_w(B \wedge C^\perp)}{p_w(C^\perp)} = \frac{\mathrm{Tr}(W(B \wedge C^\perp))}{\mathrm{Tr}(WC^\perp)} = 0$$

$$p_w(A \wedge B|C^\perp) = \frac{p_w(A \wedge B \wedge C^\perp)}{p_w(C^\perp)} = \frac{\mathrm{Tr}(W(A \wedge B \wedge C^\perp))}{\mathrm{Tr}(WC^\perp)} = 0$$

Equations (7)–(10) are satisfied by

$$1 = p_w(A \wedge B|C) = p_w(A|C)p_w(B|C) = 1$$

$$0 = p_w(A \wedge B|C^\perp) = p_w(A|C^\perp)p_w(B|C^\perp) = 0$$

$$1 = p_w(A|C) > p_w(A|C^\perp) = 0$$

$$1 = p_w(B|C) > p_w(B|C^\perp) = 0$$

So we have found a common cause C which can be regarded as the common cause of the correlation (6), and this was to be proven. ■

We can see from the proof that there is no restriction on the choice of e in $A \wedge B$. If $\dim(A \wedge B) > 1$, then there are uncountably many possible C which satisfy (7)–(10). So the common cause determined by the definition above is in general not unique.

4. SECOND GENERALIZATION

Let (i) $P(H)$ be a Hilbert lattice and W be a pure state determined by the unit vector w. For the projections E and F in the lattice let (ii) the conditional probability of E given F in a state W be defined in the following way:

$$p_w(E|F) = \frac{\mathrm{Tr}(FWFE)}{\mathrm{Tr}(FWF)}$$

The motivation of this definition comes from the theory of measurement. If we carry out a measurement of an observable represented by the projection F in a pure state W, then the state transforms as follows:

$$W \mapsto \frac{FWF}{\mathrm{Tr}(FWF)}$$

It can be easily seen that the new state is pure again. We introduce the following notation for the new pure state: $W_F \equiv FWF/\mathrm{Tr}(FWF)$. The $W \mapsto W_F$ transformation can be regarded as the 'renormalized projection' of the state W onto the subspace *RanF*. This rule is due to Lüders (1951; Bub, 1979). Using the above notation, we are able to define the common cause in terms of this new conditional probability:

Let $A, B \in P(H)$ and let there be a correlation between A and B in the state W, i.e.,

$$p_w(A \wedge B) > p_w(A)p_w(B) \tag{11}$$

Definition 3. An event C is said to be the *common cause* of the correlation between A and B if the events A, B, and C satisfy the following relations:

$$\mathrm{Tr}(W_C(A \wedge B)) = Tr(W_C A)\, \mathrm{Tr}(W_C B) \tag{12}$$

$$\mathrm{Tr}(W_{C^\perp}(A \wedge B)) = Tr(W_{C^\perp} A)\, \mathrm{Tr}(W_{C^\perp} B) \tag{13}$$

$$\mathrm{Tr}(W_C A) > \mathrm{Tr}(W_{C^\perp} A) \tag{14}$$

$$\mathrm{Tr}(W_C B) > \mathrm{Tr}(W_{C^\perp} B) \tag{15}$$

In this second quantum case we cannot prove the general existence of a common cause for all correlating events, but one can easily see also in this case how the requirement for a correlation restricts the possible arrangements of the system, and how these arrangements favor the presence of a common cause.

Now we show that not even this definition leads to a unique common cause. Let $P(H_3)$ be the projection lattice of the three-dimensional real Hilbert space H_3 with the basis $\{x, y, z\}$ (see Fig. 2). Let *RanA* be the line x and *RanB* be the plane xy. Let w be in the plane xz meeting with x at an angle β, where $\beta \in (0, \pi/2)$.

Since $0 < p_w(A \wedge B) = p_w(A) = p_w(B) < 1$ for all $\beta \in (0, \pi/2)$, so $p_w(A \wedge B) > p_w(A)p_w(B)$, i.e., there is a correlation between A and B. Now let us pick out the two new state vectors: w_C and $w_{C^\perp}$. Let w_C stand in direction x, $w_{C^\perp}$ in direction z. So the conditional probabilities are the following:

$$Tr(W_C(A \wedge B)) = 1, \qquad \mathrm{Tr}(W_C A) = 1, \qquad \mathrm{Tr}(W_C B) = 1$$

$$\mathrm{Tr}(W_{C^\perp}(A \wedge B)) = 0, \qquad \mathrm{Tr}(W_{C^\perp}\mathrm{A}) = 0, \qquad \mathrm{Tr}(W_{C^\perp}\mathrm{B}) = 0$$

which satisfy (12)–(15). However, the state vectors w_C and $w_{C^\perp}$ do not determine the projections C and $C^\perp$ uniquely. Our two different choices, for example, are the following: let $C_1 = A$, i.e., the line x; or let $C_2 = B$, i.e., the plane xy. Both choices lead to the same new state vectors w_C and $w_{C^\perp}$,

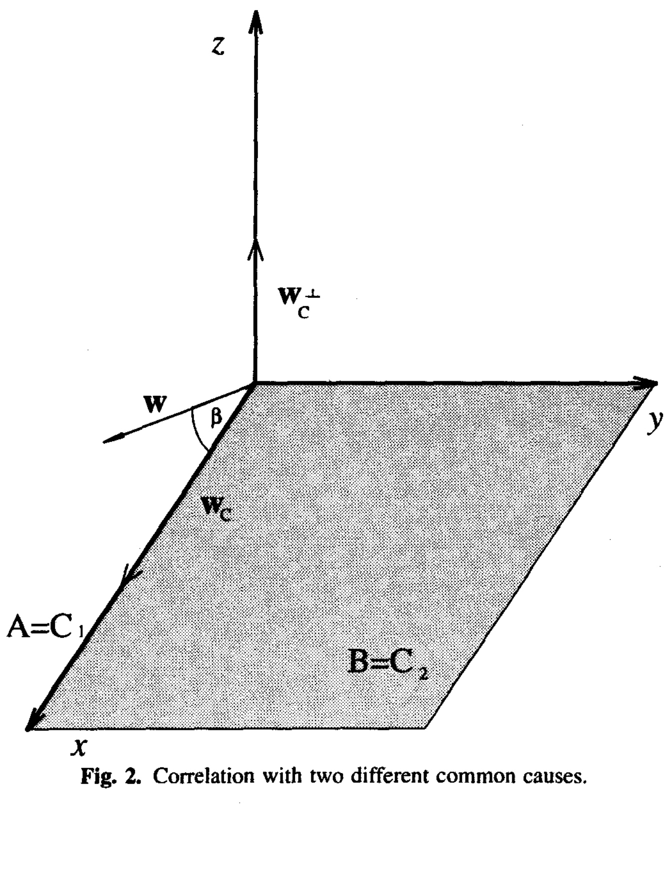

Fig. 2. Correlation with two different common causes.

which satisfy the prescribed requirements for the common cause. So not even our second common cause is unique.

REFERENCES

Bub, J. (1979). Conditional probabilities in non Boolean possibility structures, in *The Logico Algebraic Approach of Quantum Mechanics, Vol. II. Contemporary Consolidation*, C. A. Hooker, ed., Reidel, Dordrecht, Holland.

Hofer-Szabó, G. (1997a). Reichenbach's common cause definition on Hilbert lattices, *International Journal of Theoretical Physics*, submitted.

Hofer-Szabó, G. (1997b). Can Reichenbach's common cause definition be generalized to noncommutative event structures? *Foundations of Physics*, submitted.

Lüders, G. (1951). Über die Zustandsänderung durch den Messprozess, *Annalen der Physik*, **8**, 332–328.

Reichenbach, H. (1956). *The Direction of Time*, University of California Press, Berkeley, California.